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Theory of radicals for hereditarily artinian rings

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I. Introduction

As is well known, a radical-semisimple theory can be built up in every universal class of rings, but in most of the cases the universal class considered is that of all associative rings or at least another *variety* of rings. So it seems to be reasonable to have a better look at radical-semisimple theory developed in a class which is *not a variety*. In the present paper we shall consider the class **K** of all artinian rings with artinian Jacobson radical i.e. of every artinian ring each ideal of which is artinian. The rings of **K** will be called *hereditarily artinian rings*. The class **K** does not admit infinite direct sums and subrings of **K**-rings are not necessarily **K**-rings, though **K** is hereditary, homomorphically closed, and closed under extensions either. The structure of **K**-rings has been described in [2]. In the present paper we shall develop the general radical and semisimple theory in the category **K**, we shall characterize the radical and semisimple classes by certain algebraic properties and shall give explicitely all the radical and semisimple classes of **K**. Among others it will be proved that a subclass of **K** is a semisimple class iff it is hereditary and closed under extensions, further a subclass **R** is a radical class iff **R** is homomorphically closed, closed under extensions, and contains the zeroring $Z(p^\infty)$ whenever $Z(p)\in\mathbf{R}$. Also all N-radicals in **K** are determined. Since **K** is not a variety, connections among algebraic properties are different from those in a variety. For instance, in **K** every hereditary radical class is a homomorphically closed semisimple class, but the converse statement is not true. Let us remind that in a ring variety the situation is just the opposite: homomorphically closed semisimple classes are always subvarieties and strongly hereditary radical classes, but not conversely (cf. e.g. [5] Theorem 34.1 and Corollary 32.2).

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2. Preliminaries

A subclass $\mathbf{R}$ of $\mathbf{K}$ is called a *radical class* if it satisfies the following conditions:

(R_1) if $A \in \mathbf{R}$, then every non-zero homomorphic image B of A has a non-zero accessible subring C in $\mathbf{R}$;

(R_2) if every non-zero homomorphic image B of a ring $A \in \mathbf{K}$ has a non-zero accessible subring C in $\mathbf{R}$, then $A \in \mathbf{R}$.

Dualizing the definition of radical classes we get that of semisimple classes. A subclass $\mathbf{S}$ of $\mathbf{K}$ is said to be a *semisimple class,* if it satisfies the conditions:

(S_1) if $A \in \mathbf{S}$, then every non-zero accessible subring B of A has a non-zero homomorphic image C in $\mathbf{S}$;

(S_2) if every non-zero accessible subring B of a ring $A \in \mathbf{K}$ has a non-zero homomorphic image C in $\mathbf{S}$, then $A \in \mathbf{S}$.

Let us consider the class functions $\mathscr{U}$ and $\mathscr{S}$ defined by

$$\mathscr{U}\mathbf{P} = \{A \in \mathbf{K} \mid A \text{ has no non-zero homomorphic image in } \mathbf{P}\}$$

and

$$\mathscr{S}\mathbf{Q} = \{A \in \mathbf{K} \mid A \text{ has no non-zero ideal in } \mathbf{Q}\}.$$

As has been shown in [1] Theorem 1 in the framework of a more general theory, the class functions $\mathscr{U}$ and $\mathscr{S}$ establish a Galois-correspondence between hereditary and homomorphically closed subclasses of $\mathbf{K}$, and the closed subclasses are exactly the semisimple and radical classes, respectively. Thus, if $\mathbf{P}$ is a hereditary subclass of $\mathbf{K}$ and $\mathbf{Q}$ is a homomorphically closed subclass of $\mathbf{K}$, then $\mathscr{U}\mathbf{P}$ is a radical class and $\mathscr{S}\mathbf{Q}$ is a semisimple class, moreover, for each radical class $\mathbf{R}$ and semisimple class $\mathbf{S}$ we have $\mathbf{R} = \mathscr{U}\mathscr{S}\mathbf{R}$ and $\mathbf{S} = \mathscr{S}\mathscr{U}\mathbf{S}$. *Replacing the notion "accessible subring" by that of "ideal" in conditions* (R_1), (R_2), (S_1) *and* (S_2) *we get again the definitions of radical and semisimple classes* (cf. for instance [5] Theorems 10.6 and 10.7). It is easy to check that for every radical class $\mathbf{R}_A$ and semisimple class $\mathbf{S}_A$ of the class $\mathbf{A}$ of all associative rings, the intersections $\mathbf{R}_A \cap \mathbf{K}$ and $\mathbf{S}_A \cap \mathbf{K}$ yield radical and semisimple classes in $\mathbf{K}$, respectively. Further, if $\mathbf{R}$ is a radical class in $\mathbf{K}$ and $\mathscr{L}\mathbf{R}$ denotes the lower radical class of $\mathbf{A}$ generated by $\mathbf{R}$ that is

$$\mathscr{L}\mathbf{R} = \left\{ A \in \mathbf{A} \,\middle|\, \begin{array}{l} \text{every non-zero homomorphic image of } A \\ \text{has a non-zero accessible subring in } \mathbf{R} \end{array} \right\}$$

then $\mathbf{R} = \mathscr{L}\mathbf{R} \cap \mathbf{K}$ holds. Similarly, if $\mathbf{S}$ is a semisimple class of $\mathbf{K}$ and

$$\mathscr{M}\mathbf{S} = \left\{ A \in \mathbf{S} \,\middle|\, \begin{array}{l} \text{every non-zero accessible subring of } A \\ \text{has a non-zero homomorphic image in } \mathbf{S} \end{array} \right\}$$

is the semisimple class of $\mathbf{A}$ generated by $\mathbf{S}$, then $\mathbf{S} = \mathscr{M}\mathbf{S} \cap \mathbf{K}$ is valid.

All these considerations are valid not only for **K**, but for any hereditary and homomorphically closed subclass of **A**.

For further details of the radical theory we refer to [5].

Let A be a ring. We shall denote the additive group of A by $(A, +)$, the Jacobson radical of A by $\mathrm{J}(A)$, and the ring of all $n\times n$ matrices over A by A_n, respectively. The symbol $\oplus$ stands for ring theoretic direct sum. For any prime p, $Z(p^k)$ denotes the cyclic group of order p^k and also the zeroring on this group, $Z(p^\infty)$ the quasicyclic p-group and also the zeroring on this group.

In the following we use the results of [2]. The main result of [2] (Satz 3) states that A is hereditarily artinian iff

$$A = S^{(1)}_{n_1} \oplus \dots \oplus S^{(k)}_{n_k} \oplus A^* \tag{D}$$

where each $S^{(i)}_{n_i}$ is the matrix ring over an infinite division ring $S^{(i)}$ and A^* is a strong artinian ring, i.e. $(A^*, +)$ satisfies the minimum condition on subgroups. As is well known, A^* is a torsion ring and

$$A^* = A(p_1) \oplus \dots \oplus A(p_l)$$

where the $A(p_i)$-s, the so called p_i-components of A^*, are p_i-rings for distinct primes p_i. $(A(p_i), +)$ is a direct sum of a finite group and finitely many copies of $Z(p^\infty)$ lying in the annihilator of $A(p_i)$ (and thus of A).

3. Semisimple classes of K

A class **H** is said to be *hereditary,* if $I\lhd A\in \mathbf{H}$ implies $I\in\mathbf{H}$. We say that a class **E** is *closed under extensions,* if $B\lhd A$, $B\in\mathbf{E}$ and $A/B\in\mathbf{E}$ implies $A\in\mathbf{E}$.

Theorem 1. *A subclass* **S** *of* **K** *is a semisimple class in* **K** *iff*

(i) **S** *is hereditary, and*

(ii) **S** *is closed under extensions.*

Proof. In view of [5] Theorem 30.1 it suffices to show that conditions (i) and (ii) imply (S_1) and (S_2). The validity of (S_1) follows immediately from (i).

Next, take a ring $A\in\mathbf{K}$ satisfying the requirements of condition (S_2). We shall prove that $A\in\mathbf{K}$. For this end let us consider the ideal

$$I = \bigcap_\alpha (M_\alpha \lhd A \mid A/M_\alpha\in \mathbf{S}).$$

Since A is artinian, the ideal I can be represented as a finite intersection $I=\bigcap\limits_{i=1}^{n} M_i$. By induction we exhibit $A/I\in\mathbf{S}$. For $n=1$ the statement is trivial. Assuming

$A/\bigcap_{i=1}^{n-1} M_i \in \mathbf{S}$ we get

$$(*) \qquad A/\bigcap_{i=1}^{n} M_i/(M_n/\bigcap_{i=1}^{n} M_i) \cong A/M_n \in \mathbf{S}$$

and

$$M_n/\bigcap_{i=1}^{n} M_i \cong \Big(M_n+\bigcap_{i=1}^{n=1} M_i\Big)\Big/\bigcap_{i=1}^{n-1} M_i \lhd A/\bigcap_{i=1}^{n-1} M_i.$$

By (i) and the hypothesis it follows that $M_n/\bigcap_{i=1}^{n-1} M_i \in \mathbf{S}$, hence $(*)$ and (ii) imply $A/I \in \mathbf{S}$.

The proof will be done if we show $I=0$. Suppose $I \neq 0$ and consider the ideal

$$J = \bigcap_{\beta} (K_\beta \lhd I \mid I/K_\beta \in \mathbf{S})$$

of I. We want to see that J is an ideal of A, too. Taking into account that $\mathbf{K}$ is hereditary and so I also is artinian, by arguments similar to those used in proving $A/I \in \mathbf{S}$, we get $I/J \in \mathbf{S}$. Choose an arbitrary element $a \in A$ and define the mapping

$$\varphi : J \to (aJ+J)/J$$

by $\varphi(x) = ax+J$ for all $x \in J$. By [5] Proposition 5.1 φ maps J homomorphically onto the ideal $(aJ+J)/J$ of I/J. Since $I/J \in \mathbf{S}$, condition (i) implies

$$J/\operatorname{Ker}\varphi \cong (aJ+J)/J \in \mathbf{S}$$

where

$$\operatorname{Ker}\varphi = \{y \in J \mid ay \in J\}.$$

We claim that $\operatorname{Ker}\varphi$ is an ideal of I, too. Suppose that $y \in \operatorname{Ker}\varphi$ and $i \in I$. Then

$$a(iy) = (ai)y \in J \quad \text{and} \quad a(yi) = (ay)i \in J,$$

since $y \in \operatorname{Ker}\varphi$. Thus $\operatorname{Ker}\varphi$ is an ideal of I. Now

$$I/\operatorname{Ker}\varphi\,/(J/\operatorname{Ker}\varphi) \cong I/J \in \mathbf{S}$$

holds and since $J/\operatorname{Ker}\varphi \in \mathbf{S}$, condition (ii) implies $I/\operatorname{Ker}\varphi \in \mathbf{S}$. Hence $J = \bigcap_\beta K_\beta \subset \operatorname{Ker}\varphi$ and it follows that $(aJ+J)/J \cong J/\operatorname{Ker}\varphi = 0$. Thus $aJ \subset J$ holds for every $a \in A$. We get similarly $Ja \in A$, and so J is an ideal of A. Moreover, applying (ii) by $I/J \in \mathbf{S}$ and

$$A/J/(I/J) \cong A/I \in \mathbf{S}$$

we have $A/J \in \mathbf{S}$. Hence $I \subset J$ follows. But on the other hand by the assumption upon A the non-zero ideal I of A has a non-zero homomorphic image I/K_β in $\mathbf{S}$, hence $J \subset I$ and $J \neq I$ follows, a contradiction. Thus $I=0$ and the proof is complete.

The proof of Theorem 1 is a modified version of the proof given in [3] for characterizing semisimple classes of associative or alternative rings as hereditary classes being closed under extensions and subdirect sums. Working in the category $\mathbf{K}$, we could eliminate the requirement of being closed under subdirect sums.

Theorem 2. *Let* $\mathbf{S}$ *be a semisimple class of* $\mathbf{K}$. *Then there exist*

(1) *a set* $P(\mathbf{S})$ *of primes and for each prime* $p_i \in P(\mathbf{S})$ *a semisimple class* $\mathbf{L}_{p_i}(\mathbf{S})$ *of strong artinian* p_i*-rings containing all finite nilpotent* p_i*-rings,*

(2) *a class* $\mathbf{H}_1(\mathbf{S})$ *of matrix rings over infinite division rings,*

(3) *a class* $\mathbf{H}_2(\mathbf{S})$ *of matrix rings over finite fields of characteristic* $\notin P(\mathbf{S})$ *such that* $A \in \mathbf{S}$ *iff* A *is a direct sum of rings taken from* $\cup (\mathbf{L}_{p_i}(\mathbf{S}) | p_i \in P(\mathbf{S})) \cup \mathbf{H}_1(\mathbf{S}) \cup \cup \mathbf{H}_2(\mathbf{S})$.

Conversely, if P *is a set of primes,* $\mathbf{L}_{p_i}(p_i \in P)$, $\mathbf{H}_1$, $\mathbf{H}_2$ *classes of rings as given in* (1), (2), (3), *then*

$$\mathbf{S}' = \{A \mid A \text{ is a finite direct sum of rings from } (\cup \mathbf{L}_{p_i}) \cup \mathbf{H}_1 \cup \mathbf{H}_2\}$$

is a semisimple class of $\mathbf{K}$. *Moreover if* $P = P(\mathbf{S})$, $\mathbf{L}_{p_i} = \mathbf{L}_{p_i}(\mathbf{S})$ *for every prime* $p_i \in P$ *and* $\mathbf{H}_1 = \mathbf{H}_1(\mathbf{S})$, $\mathbf{H}_2 = \mathbf{H}_2(\mathbf{S})$, *then* $\mathbf{S} = \mathbf{S}'$.

Proof. Assume that $\mathbf{S}$ is a semisimple class, that is $\mathbf{S}$ satisfies (i) and (ii). Define

$$\mathbf{H}_1(\mathbf{S}) = \left\{ \begin{array}{l} \text{all matrix rings over infinite division rings} \\ \text{occurring in the decomposition } (D) \text{ of any ring of } \mathbf{S} \end{array} \right\}$$

Also define $P(\mathbf{S})$ by $p_i \in P(\mathbf{S})$ iff there exists a ring $A \in \mathbf{S}$ with $\mathbf{J}(A(p_i)) \neq 0$. By (i) also $\mathbf{J}(A(p_i)) \in \mathbf{S}$ holds and for any $p_i \in P(\mathbf{S})$. (i) and (ii) easily yield that $\mathbf{S}$ contains every finite nilpotent p_i-ring. (If $Z(p_i^\infty)$ is contained in a ring of $\mathbf{S}$, then clearly $\mathbf{S}$ contains all nilpotent artinian p_i-rings.) Let us consider the classes

$$\mathbf{H}_2(\mathbf{S}) = \left\{ \begin{array}{l} \text{all matrix rings over finite fields of characteristic} \\ \notin P(\mathbf{S}) \text{ occurring as a direct summand of any ring of } \mathbf{S} \end{array} \right\}$$

and

$$\mathbf{L}_{p_i}(\mathbf{S}) = \left\{ \begin{array}{l} \text{all } p_i\text{-rings, } p_i \in P(\mathbf{S}), \text{ occurring as a direct summand} \\ \text{of the strong artinian part of any ring of } \mathbf{S} \end{array} \right\}$$

Since $\mathbf{S}$ satisfies (i) and (ii), so does every $\mathbf{L}_{p_i}(\mathbf{S})$.

Conversely, suppose that P, $\mathbf{L}_{p_i}(p_i \in P)$, $\mathbf{H}_1$, $\mathbf{H}_2$ are given as required. We have to show that the class $\mathbf{S}'$ defined above has properties (i) and (ii). The class $\mathbf{K}$ is closed under extensions (this follows easily from the fact that $\mathbf{K}$ is hereditarily artinian). Hence an extension A of a ring of $\mathbf{S}'$ with a ring of $\mathbf{S}'$ is contained in $\mathbf{K}$ and so the main result of [2] is applicable to A. Thus as each $\mathbf{L}_{p_i}$ satisfies (ii),we obtain $A \in \mathbf{S}'$. Again by the main result of [2] the class $\mathbf{S}'$ satisfies also (i).

The last statement of the Theorem is obvious.

In Theorem 2 $P(\mathbf{S})$, $\mathbf{H}_1$ or $\mathbf{H}_2$ may be empty. Further, let us remark that the semisimple classes of $\mathbf{K}$ are described only up to semisimple classes $\mathbf{L}_p$ of strong artinian p-rings containing all finite nilpotent p-rings. For homomorphically closed semisimple classes the characterization will be more explicit.

If A is a ring of $\mathbf{K}$, then by the Wedderburn—Artin Structure Theorem the ring $A/\mathbf{J}(A)$ is a direct sum of infinite simple rings and a uniquely determined finite ring F. The ring F will be called the *finite part* of $A/\mathbf{J}(A)$.

Proposition 3. *A semisimple class* $\mathbf{S}$ *of* $\mathbf{K}$ *is homomorphically closed iff it contains also the finite part of* $A/\mathbf{J}(A)$, *whenever* $A \in \mathbf{S}$.

Proof. Let $\mathbf{S}$ be a semisimple class. If $\mathbf{S}$ is homomorphically closed and $A \in \mathbf{S}$, then the finite part of $A/\mathbf{J}(A)$ is clearly a homomorphic image of A and therefore contained in $\mathbf{S}$.

Conversely, assume that $\mathbf{S}$ satisfies the condition imposed in the Proposition. From Theorem 2 it follows that for each $p_i \in P(\mathbf{S})$ all finite nilpotent p_i-rings are in $\mathbf{S}$ and if $Z(p_i^\infty) \in \mathbf{S}$, then all artinian nilpotent p_i-rings are in $\mathbf{S}$. Hence all factor rings of $\mathbf{J}(A)$ are in $\mathbf{S}$. Thus, if I is an ideal of A, we have

$$(I+\mathbf{J}(A))/I \cong \mathbf{J}(A)/I \cap \mathbf{J}(A) \in \mathbf{S}.$$

Applying Theorem 2 and the assumption that the finite part of $A/\mathbf{J}(A)$ is in $\mathbf{S}$, it follows that $A/\mathbf{J}(A) \in \mathbf{S}$. Since

$$A/I/((I+\mathbf{J}(A))/I) \cong A/(I+\mathbf{J}(A)),$$

we get $A/I \in \mathbf{S}$ by (ii) whenever $A/(I+\mathbf{J}(A)) \in \mathbf{S}$. But $A/(I+\mathbf{J}(A))$, as a factor ring of $A/\mathbf{J}(A)$, is isomorphic to a direct summand of $A/\mathbf{J}(A)$ and therefore by (i) contained in $\mathbf{S}$.

Theorem 4. *A homomorphically closed semisimple class* $\mathbf{S}$ *is uniquely determined by two sets* P_1, P_2 *of primes with* $P_2 \subset P_1$ *and a class* $\mathbf{H}$ *of matrix rings over division rings in the following way:* $A \in \mathbf{S}$ *iff* $A/\mathbf{J}(A)$ *is a direct sum of rings of* $\mathbf{H}$, *and* $\mathbf{J}(A)$ *is a direct sum of nilpotent artinian* p_i*-rings for* $p_i \in P_1$ *and* $\mathbf{J}(A)$ *does not contain* $Z(p_i^\infty)$ *if* $p_i \in P_2$.

Proof. Choosing $P_1 = P(\mathbf{S})$ the proof follows immediately from Theorem 2 and Proposition 3.

4. Radical classes in K

Firstly we shall characterize the radical classes of $\mathbf{K}$ by

Theorem 5. *A subclass* $\mathbf{R}$ *of* $\mathbf{K}$ *is a radical class in* $\mathbf{K}$ *iff*

(a) $\mathbf{R}$ *is homomorphically closed,*

(b) $\mathbf{R}$ *is closed under extensions,*

(c) *if* $Z(p) \in \mathbf{R}$ *for a prime* p, *then also* $Z(p^\infty) \in \mathbf{R}$.

Proof. Let $\mathbf{R}$ be a radical class in $\mathbf{K}$. Then (a) and (b) follows as in Theorems 3.2 and 3.3 in [5]. If $Z(p) \in \mathbf{R}$ for a prime p, then by (R_2) it follows immediately that $Z(p^\infty) \in \mathbf{R}$.

Conversely, let $\mathbf{R}$ be a subclass of $\mathbf{K}$ satisfying conditions (a), (b), (c). Condition (a) implies trivially the validity of $(\mathbf{R}_1)$. To show that $\mathbf{R}$ is a radical class it suffices to show that if every non-zero homomorphic image of a ring $A\in\mathbf{K}$ has a non-zero ideal in $\mathbf{R}$, then $A\in\mathbf{R}$ also holds. In view of decomposition (D) we may confine ourselves to the case $A=A(p)$ where $A(p)$ is a strong artinian p-ring. We claim that the maximal divisible ideal D of A is in $\mathbf{R}$. For $D=0$ the assertion is trivial. If $D\neq 0$, then considering the so called kernel A_0 of A (cf. [2] Satz 1), A_0 is a finite image ideal of A contained in D, furthermore, $A/A_0\cong D$. Hence D, as a homomorphic of A, contains a non-zero ideal in $\mathbf{R}$. Now (a), (c) and (b) imply $Z(p)\in\mathbf{R}$, $Z(p^\infty)\in\mathbf{R}$ and $D\in\mathbf{R}$.

Applying (a), (b) and the second isomorphism theorem, the sum of two $\mathbf{R}$-ideals of A/D is again in $\mathbf{R}$. Hence A/D being finite, it contains a maximal $\mathbf{R}$-ideal J/D. If $J\neq A$ then by assumption A/J has a non-zero $\mathbf{R}$-ideal K/J. Thus by (b) and

$$K/D/(J/D)\cong K/J$$

we obtain $K/D\in\mathbf{R}$ contradicting the maximality of J/D. Hence $A=J$ and by $A/D\in\mathbf{R}$, $D\in\mathbf{R}$ condition (b) infers $A\in\mathbf{R}$. Thus $(\mathbf{R}_2)$ holds.

Corollary 6. *Every hereditary radical class in* $\mathbf{K}$ *is a homomorphically closed semisimple class.*

This is clear by Theorems 1 and 5.

The converse statemant of Corollary 6 is, however, false. *A homomorphically closed semisimple class in* $\mathbf{K}$ *need not be a radical class.* Take, for instance, the class $\mathbf{M}$ of all finite nilpotent p-rings for a fixed prime p. $\mathbf{M}$ is a homomorphically closed semisimple class. But $\mathbf{M}$ fails to be a radical class, for $Z(p^\infty)\notin\mathbf{M}$. Thus a class $\mathbf{P}\subset\mathbf{K}$ which is homomorphically closed and closed under extensions, is not necessarily a radical class i.e. condition (c) is necessary.

Theorem 7. *Let* $\mathbf{R}$ *be a radical class in* $\mathbf{K}$. *Then there exist*

(1) *a set* $Q(\mathbf{R})$ *of primes and for every* $p_i\in Q(\mathbf{R})$ *a radical class* $\mathbf{M}_{p_i}(\mathbf{R})$ *of strong artinian* p_i*-rings which does not contain non-zero finite nilpotent rings;*

(2) *a class* $\mathbf{H}(\mathbf{R})$ *of matrix rings over division rings containing all such rings which are in* $\mathbf{M}_{p_i}(\mathbf{R})$, $p_i\in Q(\mathbf{R})$; *such that* $A\in\mathbf{R}$ *iff* $A/\mathbf{J}(A)$ *is a finite direct sum of rings of* $\mathbf{H}(\mathbf{R})$ *and* $A(p_i)\in\mathbf{M}_{p_i}(\mathbf{R})$ *for every* $p_i\in Q(\mathbf{R})$.

Conversely, if Q *is a set of primes,* $\mathbf{M}_{p_i}$ *and* $\mathbf{H}$ *are classes as required in* (1) *and* (2), *respectively, then*

$$\mathbf{R}'=\left\{A\in\mathbf{K}\;\middle|\;\begin{array}{l}A/\mathbf{J}(A)\text{ is a finite direct sum of rings}\\ \text{of }\mathbf{H}\text{ and }A(p_i)\in\mathbf{M}_{p_i},\,p_i\in Q\end{array}\right\}$$

is a radical class. Moreover, if $Q=Q(\mathbf{R})$, $\mathbf{M}_{p_i}=\mathbf{M}_{p_i}(\mathbf{R})$, $(p_i\in Q)$, $\mathbf{H}=\mathbf{H}(\mathbf{R})$, *then* $\mathbf{R}'=\mathbf{R}$.

Proof. Let $\mathbf{R}$ be a radical class in $\mathbf{K}$. Define $Q(\mathbf{R})$ by $p_i \in Q(\mathbf{R})$ iff $\mathbf{R}$ does not contain non-zero finite nilpotent p_i-rings, and define

$$\mathbf{H}(\mathbf{R}) = \left\{ \begin{array}{l} \text{all matrix rings over division rings occurring as direct summands} \\ \text{in any factor ring } A/\mathbf{J}(A),\ A \in \mathbf{R} \end{array} \right\},$$

$$\mathbf{M}_{p_i}(\mathbf{R}) = \left\{ \begin{array}{l} \text{all } p_i\text{-rings occurring as direct components in the} \\ \text{decomposition (D) of any ring } A \in \mathbf{R} \text{ for } p_i \in Q(\mathbf{R}) \end{array} \right\}.$$

Since by Theorem 5 $\mathbf{R}$ is homomorphically closed and closed under extensions, so is each $\mathbf{M}_{p_i}(\mathbf{R})$, $p_i \in Q(\mathbf{R})$. The classes $\mathbf{M}_{p_i}(\mathbf{R})$ satisfy condition (c) of Theorem 5 trivially.

Conversely, if Q is a set of primes, $\mathbf{M}_{p_i}$, $p_i \in Q$, are radical classes of p_i-rings as demanded in (1), $\mathbf{H}$ is a class of matrix rings over division rings, then define $\mathbf{R}'$ as above. Since each $\mathbf{M}_{p_i}$ is homomorphically closed, also $\mathbf{R}'$ is homomorphically closed. Condition (c) is trivially fulfilled by the definition of $\mathbf{R}'$. Let $I \lhd A$ be such that $I, A/I \in \mathbf{R}'$. Since A is artinian,

$$\mathbf{J}(A/I) = (I + \mathbf{J}(A))/I$$

holds and consequently

$$A/(I+\mathbf{J}(A)) \cong A/I/((I+\mathbf{J}(A))/I) = A/I/(\mathbf{J}(A/I))$$

is a direct sum of rings of $\mathbf{H}$, since $A/I \in \mathbf{R}'$. $A/\mathbf{J}(A)$ contains $(I+\mathbf{J}(A))/\mathbf{J}(A) \cong I/\mathbf{J}(I)$ as a direct summand. The simple direct summands of $I/\mathbf{J}(I)$ are contained in $\mathbf{H}$, since $I \in \mathbf{R}'$. The other simple direct summands of $A/\mathbf{J}(A)$ are in $\mathbf{H}$, for

$$A/\mathbf{J}(A)/((I+\mathbf{J}(A))/\mathbf{J}(A)) \cong A/(I+\mathbf{J}(A)).$$

Moreover, the classes $\mathbf{M}_{p_i}$, $p_i \in Q$, are closed under extensions, therefore $\mathbf{R}'$ is closed under extensions, too. Hence Theorem 5 yields that $\mathbf{R}'$ is a radical class of $\mathbf{K}$.

The last statement is obvious.

In Theorem 7, analogously to Theorem 2, the radical classes are determined only up to radical classes of strong artinian p_i-rings. Nevertheless, the hereditary radical classes are fully described by

Theorem 8. *A hereditary radical class* $\mathbf{R}$ *in* $\mathbf{K}$ *is uniquely determined by a set* P *of primes and a class* $\mathbf{H}$ *of matrix rings over division rings in the following way:* $A \in \mathbf{R}$ *iff* $A/\mathbf{J}(A)$ *is a direct sum of rings of* $\mathbf{H}$, *and* $\mathbf{J}(A)$ *is a finite direct sum of nilpotent artinian* p_i*-rings for* $p_i \in P$.

The statement follows immediately from Theorems 4 and 5 and from Corollary 6.

In view of Theorems 2, 5 and Corollary 6 we have

Corollary 9. *If the subclass* $\mathbf{C}$ *of* $\mathbf{K}$ *is hereditary, closed under extensions and does not contain non-zero zerorings, then* $\mathbf{C}$ *is a homomorphically closed semisimple class as well as a hereditary radical class.*

Of course, *not every radical class in* $\mathbf{K}$ *is hereditary.* We give an example for a non-hereditary radical class of $\mathbf{K}$ which does not contain zero-rings $Z(p^\infty)$. Let

N denote the ring of integers, and p a fixed prime, and consider the class

$$\mathbf{R} = \{\text{finite direct sums of copies of } N/(p) \text{ and } N/(p^2)\}.$$

One can easily check that $\mathbf{R}$ is a radical class of $\mathbf{K}$, for $\mathbf{R}$ satisfies conditions (a), (b) and (c) of Theorem 5. On the other hand $\mathbf{R}$ is not hereditary, because $(p)/(p^2) \lhd N/(p^2)$ and $(p)/(p^2) \cong Z(p) \notin \mathbf{R}$ hold. This example is the special case $Q=\{p\}$, $\mathbf{M}_p=\mathbf{R}$, $\mathbf{H}=\{N/(p)\}$ of Theorem 7.

Let us mention that in view of Theorems 4 and 8 *homomorphically closed semisimple classes and hereditary radical classes of* $\mathbf{K}$ *are not necessarily subring-hereditary classes* (e.g. whenever $\mathbf{H}$ is not subring-hereditary). We recall the fact that in a ring variety the homomorphically closed semisimple classes are always subvarieties and hence subring-hereditary classes.

5. N-radicals in K

A radical $\mathbf{R}$ is called *left hereditary,* if every left ideal of an $\mathbf{R}$-ring is also in $\mathbf{R}$. A radical $\mathbf{R}$ is said to be a *left strong radical,* if the radical $\mathbf{R}(A)$ of any ring A contains every $\mathbf{R}$-left ideal of A. Following SANDS [4], a radical $\mathbf{N}$ is called an *N-radical,* if it is left hereditary, left strong and it contains every zero-ring.

Let $\mathbf{R}$ be a left strong radical containing all zero-rings, and containing a division ring D. Take the $n\times n$ matrix ring D_n over D. Denote by ε_{ij} the matrix having $1\in D$ at the i-th j-th place and 0 at every other place. Then $D_n=\sum_{i=1}^{n} D_n\varepsilon_{ii}$, further $\varepsilon_{ii}D_n\varepsilon_{ii}\cong \cong D$. Moreover, the mapping.

$$\varphi : D_n\varepsilon_{ii} \to \varepsilon_{ii}D\varepsilon_{ii} \cong D$$

defined by

$$\varphi(x\varepsilon_{ii}) = \varepsilon_{ii}x\varepsilon_{ii} \quad (x\in D_n)$$

is a ring homomorphism onto $\varepsilon_{ii}D_n\varepsilon_{ii}$. Hence

$$D \cong D_n\varepsilon_{ii}/\operatorname{Ker}\varphi$$

and

$$\operatorname{Ker}\varphi = \{x\varepsilon_{ii}\in D_n\varepsilon_{ii} \mid \varepsilon_{ii}x\varepsilon_{ii} = 0\}$$

is a zero-ring, since $(x\varepsilon_{ii})(y\varepsilon_{ii})=x(\varepsilon_{ii}y\varepsilon_{ii})=0$ holds for every $x\varepsilon_{ii}, y\varepsilon_{ii}\in\operatorname{Ker}\varphi$. Since $\mathbf{R}$ contains all zero-rings and $D\in\mathbf{R}$, so the extension property of $\mathbf{R}$ implies $D_n\varepsilon_{ii}\in\mathbf{R}$. Taking into account that $\mathbf{R}$ is a left strong radical, we get $D_n\in\mathbf{R}$. Hence we arrived at

Proposition 10. *Let* $\mathbf{R}$ *be a left strong radical containing all zero-rings. If a division ring* D *is contained in* $\mathbf{R}$, *then every matrix ring* D_n *over* D *is also in* $\mathbf{R}$.

Next, assume that $\mathbf{R}$ is a left hereditary radical and D_n is an $n\times n$ matrix ring over a division ring D. Suppose $D_n\in\mathbf{R}$. Since $D_n=\sum_{i=1}^{n} D_n\varepsilon_{ii}$, and $\mathbf{R}$ is left hereditary, we have $D_n \varepsilon_{ii} \in \mathbf{R}$. Consider again the homomorphism φ defined above. φ maps $D_n \varepsilon_{ii}$ onto $\varepsilon_{ii} D \varepsilon_{ii} \cong D$ homomorphically. Since $\mathbf{R}$ is homomorphically closed, we have $D \in \mathbf{R}$. Thus we obtained

Proposition 11. *Let* $\mathbf{R}$ *be a left hereditary radical. If a matrix ring* D_n *over a division ring* D *is in* $\mathbf{R}$, *then also* $D \in \mathbf{R}$ *holds.*

Applying Theorem 8 and Propositions 10 and 11 we obtain a full description of N-radicals in $\mathbf{K}$.

Theorem 12. *To any N-radical* $\mathbf{N}$ *of* $\mathbf{K}$ *there belongs a class* $\mathbf{D}$ *of all division rings* D *with* $D \in \mathbf{N}$. *Conversely, any class* $\mathbf{D}$ *of division rings determines an N-radical;* $\mathbf{N}$ *is the lower radical of all zero-rings of the class* $\mathbf{K}$ *and of all matrix rings over division ring from* $\mathbf{D}$.

Comparing Theorem 12 with the situation in the variety of associative rings, there is a remarkable difference. As has been shown in [6], if an N-radical $\mathbf{N}$ of associative rings contains the Brown—McCoy radical (the upper radical of all simple rings with unity), then $\mathbf{N}$ contains all division rings. The Brown—McCoy radical fails to be left hereditary, so it is not an N-radical. (In the category $\mathbf{K}$, the Brown—McCoy radical coincides with Baer's lower radical, and it is an N-radical.) The well-known N-radicals of associative rings are Baer's lower radical, the Levitzki and the Jacobson radical and some peculiar radicals constructed artificially.

Finally we give a characterization of the class of Jacobson radical rings (i.e. all nilpotent rings) in $\mathbf{K}$.

Corollary 13. *If* $\mathbf{J}$ *is a left hereditary radical class of* $\mathbf{K}$ *such that* $\mathbf{J}$ *contains* $Z(p)$ *for every prime* p *and* $\mathbf{J}$ *does not contain division rings, then* $\mathbf{J}$ *is the class of all nilpotent rings in* $\mathbf{K}$.

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